Tubular groups, 1-relator groups and non positive curvature

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Abstract

We show (using results of Wise and of Woodhouse) that a tubular group is always virtually special (meaning that it has a finite index subgroup embedding in a RAAG) if the underlying graph is a tree. We also adapt Gardam and Woodhouse's argument on tubular groups which double cover 1-relator groups to show there exist 1-relator groups which are CAT(0) but not residually finite.

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1 Introduction

The usual notion of a discrete group G having non positive curvature is taken to be that it is CAT(0); namely it acts geometrically (properly and cocompactly by isometries) on a CAT(0) metric space. This certainly has some important consequences for the structure of the group G (see [5] Part III Chapter Γ Theorem 1.1), for instance G will be finitely presented. However Wise's example in [23] of a CAT(0) group which is not even Hopfian, thus not residually finite nor linear over any field, shows that being CAT(0) is not a sufficient condition to be well behaved in terms of abstract group theoretic properties.

In recent years though, a stronger geometric notion has emerged, namely that of acting geometrically not just on a CAT(0) space but on a CAT(0) cube complex. Indeed because of the pioneering work of Wise and Agol ([17], [24], [1]), if G is also word hyperbolic then it is *virtually special*. This means that G has a finite index subgroup H which is the fundamental group of a special non positively curved cube complex, where special is defined in terms of forbidden hyperplane pathologies (see [24] Definition 4.2). Now G is virtually special if and only if the finite index subgroup H embeds in a right angled Artin group (a RAAG), for instance see [24] Theorem 4.4 and Corollary 4.5. Note that this fact does not require the corresponding special cube complex to be compact, whereas in the word hyperbolic case mentioned above, compactness is obtained as a consequence. As it is the group theoretic consequences that most concern us here, our definition of a virtually special group will not insist on compactness (in the terminology, our groups will be virtually special but need not be virtually compact special). Indeed these group theoretic consequences will be very strong because a virtually special group G will inherit any property held by all RAAGs and preserved under taking subgroups and finite index supergroups. In particular G will be linear not just over $\mathbb{C}$ but even over $\mathbb{Z}$ (in fact there are various groups only known to be linear because they are virtually special, such as word hyperbolic free by cyclic groups as shown in [16]), as well as being virtually biorderable, virtually residually torsion free nilpotent and virtually residually finite rational solvable for instance.

This interplay between the geometric and group theoretic properties of G is therefore very powerful if G is word hyperbolic. However in this paper our focus will be on groups which, although always finitely presented (in fact all groups considered here will have a presentation of deficiency 1), will usually contain $\mathbb{Z}\times\mathbb{Z}$. The problem here is that in the absence of word hyperbolicity, a finitely presented group being virtually special neither implies nor is implied by a geometric action on a CAT(0) cube complex. To see this, first note that the famous finitely presented simple groups of Burger and Mozes act geometrically on a CAT(0) cube complex, namely a product of two trees, but their simplicity means that they cannot embed in a RAAG, nor do they have any proper finite index subgroups. Conversely there are finitely presented subgroups H of $F_2 \times F_2 \times F_2$, which is a RAAG so H will be virtually special, but which do not have the correct finiteness properties even to be CAT(0) ([5] Chapter III.Γ Section 5).

In order to investigate this further, we can restrict to classes of groups which are non word hyperbolic but whose structure gives us a better chance of determining which groups in the class behave well geometrically and which ones have good group theoretic properties. The first class we will consider here are the *tubular groups*: namely the fundamental group $\mathcal{G}(\Gamma)$ of a finite graph Γ of groups with all vertex groups isomorphic to $\mathbb{Z}^2$ and all edge groups isomorphic to $\mathbb{Z}$. Of course tubular groups can never be word hyperbolic (and indeed are never even relatively hyperbolic) but they have already been shown in other papers to possess a range of interesting behaviours. For instance Wise's non Hopfian CAT(0) group mentioned above is tubular but is not virtually special nor acts geometrically on a CAT(0) cube complex. Indeed in [25] Wise showed that only a very few tubular groups act geometrically on a CAT(0) cube complex ([25] Corollary 5.10) and these few groups are all virtually special ([25] Corollary 5.9). Furthermore a virtually special tubular group will be CAT(0) (by [25] Lemma 4.4) but certainly not vice versa.

In particular if we want our tubular group G to be well behaved from a group theoretic point of view then showing it is CAT(0) will not be enough, but showing it is virtually special certainly will be. Thus in Section 2 we look for tubular groups which are virtually special but which do not act geometrically on a CAT(0) cube complex. The downside here is that confirming directly that a group G is virtually special can be very involved, because one usually needs Wise's machinery to show not only that G has a finite index subgroup H which is the fundamental group of a non positively curved cube complex but also that the complex is special. However Woodhouse introduces in [28] a technique specifically for tubular groups which is used to show that a tubular group is virtually special if and only if it acts freely on some locally finite CAT(0) cube complex. This is achieved by adapting Wise's equitable sets condition in [25] (which is equivalent to the group having a free action on some CAT(0) cube complex) and imposing extra constraints on these equitable sets which imply that the group is virtually special. Therefore it seems appropriate to see for which tubular groups this result can be made to work. We show in Theorem 2 that Woodhouse's criterion is satisfied for any tubular group defined by $(\mathcal{G}, \Gamma)$ where Γ is a tree. Thus this gives a new class of virtually special groups and we can conclude that they all possess our strong group theoretic properties above. In particular these groups are all linear over $\mathbb{Z}$ even though linearity of this family was not previously known.

We finish in Section 3 by looking at 1-relator groups. Again we are interested in the family of non word hyperbolic 1-relator groups (it is open as to whether every word hyperbolic 1-relator group is virtually special or acts geometrically on a CAT(0) cube complex or even is CAT(0)). Here we only examine 2-generator 1-relator groups, though we note the very recent work in [20] which, on combining Theorem 1.3, Conjecture 1.6, Corollary 1.7 and Conjecture 1.8, would say that a 1-relator group with at least 3 generators is either word hyperbolic or it is hyperbolic relative to a 2-generator 1-relator group.

A well known source of 1-relator groups with bad behaviour, both group theoretic and geometric, are the famous *Baumslag - Solitar groups* $BS(m,n) = \langle a,b|ba^mb^{-1} = a^n\rangle$. More precisely, those groups where $|m| = |n|$ have $F_k \times \mathbb{Z}$ as a finite index subgroup and so are actually well behaved in terms of non positive curvature (though they are obstructions to a group being word hyperbolic). However those Baumslag - Solitar groups where $|m| \neq |n|$ act as obstructions even for non positively curved behaviour and we will refer to them as unbalanced or non Euclidean Baumslag - Solitar groups. Indeed a group containing any subgroup isomorphic to an unbalanced Baumslag - Solitar group cannot be CAT(0), let alone act geometrically on a CAT(0) cube complex, nor can it be linear over $\mathbb{Z}$ and it will certainly not be virtually special. This then begs the natural question: given a 1-relator group with no unbalanced Baumslag - Solitar subgroup, must it be well behaved geometrically or group theoretically?

This was open until the construction in [13] of Gardam and Woodhouse which obtained a very close relationship between certain 2-generator 1-relator groups and certain tubular groups. More precisely, they consider the particular class of tubular groups

$$G_{p,q} = \langle a,b,s,t|[a,b] = 1, sa^qs^{-1} = a^pb, ta^qt^{-1} = a^pb^{-1}\rangle$$

for p,q positive integers which (when $p > q$) are known as the snowflake groups introduced in [2]. In [13] it is shown that the group $G_{p,q}$ is an index 2 subgroup of a 2-generator 1-relator group $R_{p,q}$. This allows them to transfer various group theoretic and geometric properties (or the lack thereof) from the tubular group to the 1-relator group. In particular they show that for $p \geq q$ the resulting group $R_{p,q}$ is not CAT(0). However $R_{p,q}$ never contains an unbalanced Baumslag - Solitar subgroups, so this is the first example of a 1-relator group which is not well behaved geometrically even though it contains no unbalanced Baumslag - Solitar subgroups.

In Section 3 we look at these families $R_{p,q}$ and $G_{p,q}$ but vary the parameters so that $q > p$, whereupon [12] established that $R_{p,q}$ is a CAT(0) group. This allows us in Theorem 4 to give the first examples of 1-relator groups which are CAT(0) but not residually finite. We do this by showing that when $p = 1$ and $q \geq 3$ is odd, the resulting tubular subgroup $G_{p,q}$ is non Hopfian using a very similar argument to Wise in [23], which itself is an adaptation of the original argument by Baumslag and Solitar. Consequently $G_{p,q}$ and $R_{p,q}$ are not residually finite. In fact in Theorem 6 we adapt this argument, along with a result in [19], to work out exactly when the group $G_{p,q}$ (equivalently $R_{p,q}$) is residually finite (here we can take p, q to be any non zero integers): namely when q divides $2p$. Thus on regarding "well behaved geometrically" to mean CAT(0) and "well behaved group theoretically" to mean residually finite, we have 1-relator groups $R_{p,p}$ which are well behaved group theoretically but not geometrically, 1-relator groups $R_{p,3p}$ which are well behaved geometrically but not group theoretically, and (for $p > 3$) 1-relator groups $R_{p,p-1}$ which are neither, all without unbalanced Baumslag - Solitar subgroups. This construction also allows us to answer Question 20 in [7], namely (for p even) the 2-generator 1-relator group $R_{p,p/2}$ is residually finite but has no finite index subgroup which is an ascending HNN extension of a finitely generated free group.

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2 Virtually special tubular groups from trees

If we are given a tubular group $\mathcal{G}(\Gamma)$ where Γ is a tree then we build our group using repeated amalgamations over $\mathbb{Z}$ and it is straightforward to see, for instance by applying [5] Chapter II Corollary 11.19, that our tubular group will be CAT(0). This is taken further in [25] which establishes other results for tubular groups. The key concept there is that of an equitable set which we now define:

Definition 1 *An equitable set for the tubular group $(\mathcal{G}, \Gamma)$ is a choice at each vertex v in Γ of a finite number of closed curves $c_1^{(v)}, \ldots, c_{m(v)}^{(v)}$ on a torus $S^1 \times S^1$ placed at v, which therefore can be regarded as elements of $\pi_1(S^1 \times S^1) = G_v \cong \mathbb{Z}^2$, so that on taking any edge e with endpoints $e_\pm$ and edge group $\langle z_e \rangle$ embedding in G_{e_+} as $z_{e_+} \in \mathbb{Z}^2 \setminus \{(0,0)\}$ and in G_{e_-} as z_{e_-},* *then the sum of the (algebraic, unsigned) intersection numbers*

$$\#[c_1^{(e_+)}, z_{e_+}] + \ldots + \#[c_{m(e_+)}^{(e_+)}, z_{e_+}]$$

at e_+ is equal to the sum

$$\#[c_1^{(e_-)}, z_{e_-}] + \ldots + \#[c_{m(e_-)}^{(e_-)}, z_{e_-}]$$

of intersection numbers at the other end. It is also required as part of this definition that the elements at each vertex of the equitable set are not all parallel.

The importance of this definition of Wise is his result in [25] stating that the tubular group $\mathcal{G}(\Gamma)$ defined by $(\mathcal{G}, \Gamma)$ acts freely on a CAT(0) cube complex (which might well be infinite dimensional and/or not locally finite) if and only if we can find an equitable set for $(\mathcal{G}, \Gamma)$. We summarise here how this equitable set is used: first form a graph of spaces X with fundamental group $\mathcal{G}(\Gamma)$ by taking each vertex space to be a torus $S^1 \times S^1$ and at each edge e with endpoints $e_\pm$ we have an edge space which is a copy of the cylinder $S^1 \times [-1, 1]$. We then attach these two end circles S^1 to the curves $z_{e_\pm}$, where we regard z_{e_+} as a closed curve lying in the torus vertex space at e_+ and similarly z_{e_-} at e_-. The condition above on intersection numbers is used to create a bijection between the intersection points of z_{e_+} with the equitable set at the vertex e_+ and those of z_{e_-} with the equitable set at e_-. Having defined this bijection, one joins each pair of corresponding intersection points by an arc running along the edge cylinder from the vertex space at e_+ to that at e_-. On doing this over all edges in the graph Γ, one has a disjoint union of copies of S^1, one for each element in the equitable set, with these circles joined by various arcs to form some finite graph Ω which need not be connected. However each connected component of this graph has an immersion in our graph of spaces X, though this need not be an embedding, and its image in X is called an immersed wall. We can then take some copy $\tilde{\Lambda}$ of the universal cover of Λ which embeds in the universal cover $\tilde{X}$ of X. By varying over the different lifts of Λ and over all such Λ, one obtains a wallspace to which Sageev's construction of a CAT(0) cube complex applies.

Here we want to show that there are a range of tubular groups which are virtually special, in addition to those which act properly and cocompactly on a CAT(0) cube complex. Now one might think that in order to show a group is virtually special, such a geometric action has to be found for the group anyway. Whilst this is generally true in the word hyperbolic case, we note that in the case of fundamental groups of non hyperbolic 3-manifolds, the papers [15] and [22] (for graph manifolds and mixed 3-manifolds respectively) show that the fundamental groups of such manifolds can be virtually special but not virtually compact special. This phenomenon also occurs for many tubular groups: on the one hand, the characterisation (Wise's Corollary 5.10 in [25]) of those tubular groups which act geometrically on a CAT(0) cube complex is very restrictive, but we also have a result of Woodhouse in [28] which allows us to obtain tubular groups that are virtually special without finding (or even there existing) such a geometric action. We now review this result in some more detail.

First of all, if a closed curve c in our equitable set is not primitive, so that it is of the form $n(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ with $n \geq 2$ and a,b coprime, then in the Wise result we can either regard c as a primitive curve traversed n times, or as n parallel disjoint primitive curves. In the Woodhouse work the second approach is taken and we will do that here too.

In order to satisfy Woodhouse's virtually special criterion, we will be taking an equitable set consisting only of primitive elements (which can be ensured by the comment above) which is also *fortified*, which means that every time we inject an edge group $\langle z_e \rangle$ into a vertex group G_{e_+} (or G_{e_-}), the image z_{e_+} of z_e in the vertex space at e_+ is parallel to something in the equitable set at e_+, or alternatively there is at least one element c in this equitable set such that the intersection number $\#[c, z_{e_+}] = 0$.

A primitive, fortified equitable set then gives rise to primitive, fortified immersed walls using the Wise construction as described above, from which we obtain our graph Ω. Each circle in Ω embeds in a vertex space and so can be thought of as "lying over" the relevant vertex v of the graph Γ. Similarly the arcs of Ω come from taking an edge e of Γ and choosing a bijection between the intersection points on either side of e, then adding an edge in Ω between each of these pairs of intersection points, so we can think of such an arc as lying over the edge e.

Now let the connected components of Ω be $S_1, \ldots, S_k$. The criterion that we will now use is [28] Proposition 4.8, which states that if the images $\Lambda_1, \ldots, \Lambda_k$ of $S_1, \ldots, S_k$ in X are all primitive, fortified, undilated immersed walls then the tubular group $G = \pi_1(\mathcal{G})$ is virtually special, with only the condition of being *undilated* left to explain. We describe this as follows: suppose we have a directed edge path $e_1, \ldots, e_n$ in the graph Ω, or rather the graph obtained from Ω by shrinking each circle to a point. The dilation function from such edge paths to $\mathbb{Q}^*$ is calculated as follows: we start with value 1 at the vertex $(e_1)_-$ in Ω, which corresponds to an element of our equitable set, and then as we traverse the edge e_1 to arrive at the vertex $(e_1)_+$, we multiply our value by the ratio of intersection numbers $\#[(e_1)_-, z_{(\overline{e_1})_-}]/\#[(e_1)_+, z_{(\overline{e_1})_+}]$. Here $\overline{e_1}$ is the edge in Γ lying below the edge $e_1 \in \Omega$ and $z_{(\overline{e_1})_\pm}$ are the respective inclusions at each end of the generator $z_{(\overline{e_1})}$ of the edge group for $\overline{e_1}$ in Γ. We then multiply our value by $\#[(e_2)_-, z_{(\overline{e_2})_-}]/\#[(e_2)_+, z_{(\overline{e_2})_+}]$ as we cross the edge e_2 in Ω and so on, thus obtaining the value of our dilation function for this edge path when we arrive at the final vertex $(e_n)_+$. We then say that Ω has undilated immersed walls if the dilation function of every closed path in Ω is 1.

Now although our original graph Γ is a tree, the components of Ω need not be. Even if they are, this does not ensure that Ω has undilated immersed walls, as demonstrated in [27] Example 4.6.2 where the tubular group $\langle a, b, c, d | [a, b], [b, c], [c, d]\rangle$ is taken, along with an equitable set and pairing of intersection points to create a graph Ω which is in fact connected but where there exists a dilated immersed wall. As this group is clearly a RAAG, we see that we have to take care both in providing an appropriate equitable set and in choosing suitable bijections between intersection points in order to conclude a group is virtually special using this criterion.

However the method that we will adopt in order to obtain undilated immersed walls is the following: suppose that each immersed wall Λ_i $(1 \leq i \leq k)$ given by the image in X of the connected subgraph S_i of Ω is actually an embedding of Ω_i. Then any two lifts $\tilde{\Lambda}_i$ and $\tilde{\Lambda}'_i$ say of Λ_i in the universal cover $\tilde{X}$ must be disjoint, as otherwise an intersection would project down to a self intersection of Λ_i under the corresponding map $\pi : \tilde{X} \to X$. But by [26] Proposition 4.6, if we have some dilated wall $\tilde{\Lambda}$ in our wallspace then there are translates of $\tilde{\Lambda}$ which intersect.

Theorem 2 *Let G be a tubular group obtained from the graph of groups $(\mathcal{G}, \Gamma)$ where Γ is a tree. Then there exists a fortified, primitive, equitable set for $(\mathcal{G}, \Gamma)$ such that all immersed walls obtained from this equitable set are embedded. Consequently G has a finite index subgroup that embeds in a RAAG.*

Proof. The construction of our equitable set can be summarised as follows: at each vertex in turn, examine the images of all edge groups embedding in this vertex group and begin by taking the primitive element in each parallelism class of these images. These must be included if we are to obtain a fortified and primitive equitable set for $(\mathcal{G}, \Gamma)$, though we will also add some other arbitrary primitive elements to some vertices until these sets at each vertex are the same size. At this point we have not considered intersection numbers at all and so have made no attempt to create an equitable set for $(\mathcal{G}, \Gamma)$, let alone one with embedded walls. However both of these conditions will be achieved by taking many parallel copies of the elements at each vertex. The number of copies will be chosen when we add a new vertex of Γ at each stage, by using the intersection number with the new edge group to balance out the copies of these sets at both ends of our edge so that all intersection numbers are equal. However we will need to backtrack and take more copies of our sets at the vertices which have already been added.

In more detail: let v_0 be the vertex of Γ with the most parallelism classes of edge groups (in fact [25] Corollaries 5.9 and 5.10 tell us that if this number is at most 2 then G is already virtually special, so we can assume that we have at least 3 to avoid small cases). Say there are m (≥ 3) classes at v_0 and take $\mathbf{x}_1^{(0)}, \ldots, \mathbf{x}_m^{(0)}$ to be the primitive elements in each parallelism class. Now label the other vertices consecutively using a breadth based order with v_0 at the root. Then at each other vertex v_i take a primitive element for each parallelism class of edge groups in the vertex group G_{v_i}, which we will label as $\mathbf{x}_1^{(i)}, \ldots, \mathbf{x}_{m_i}^{(i)}$ and where we have $m_i \leq m$. Next we add arbitrary new primitive elements at each vertex until we have m of them, thus obtaining $\mathbf{x}_1^{(i)}, \ldots, \mathbf{x}_m^{(i)}$ at each vertex v_i where now m is independent of i and these m primitive elements are all distinct.

We now proceed through the tree, vertex by vertex, to build our equitable set out of repeated copies of these elements. We will end up by building m subgraphs Ω_i, which need not be connected but which will embed in X such that Ω is the disjoint union of $\Omega_1, \ldots, \Omega_m$. First take the base v_0 and an adjacent vertex v_1, connected by the edge e. On taking as usual z_e to be a generator of the edge group, with inclusions z_{e_-} in the vertex v_0 and z_{e_+} at v_1, our first priority is to examine the intersection numbers

$$\begin{aligned} &i_1 = \#[\mathbf{x}_1^{(0)}, z_{e_-}], \quad \ldots \quad, i_m = \#[\mathbf{x}_m^{(0)}, z_{e_-}] \qquad \text{and} \\ &j_1 = \#[\mathbf{x}_1^{(1)}, z_{e_+}], \quad \ldots \quad, j_m = \#[\mathbf{x}_m^{(1)}, z_{e_+}]. \end{aligned}$$

In each list of numbers, exactly one is zero and the rest are positive because at each vertex the primitive element parallel to this edge group was included, but no other element parallel to it was included. Thus all other numbers in each list are strictly positive, so here we renumber to make $i_1 = j_1 = 0$. We now start to create the graph Ω_1 by placing a copy of S^1 in the vertex space at v_0 corresponding to the element $\mathbf{x}_1^{(0)}$, a circle above v_1 for the element $\mathbf{x}_1^{(1)}$ and no edge between them. Moving now to i_2 and j_2, there is no a priori relationship between them, so no reason to assume $i_2 = j_2$ which would enable the construction of an equitable set. But we are free to have repeated copies of these elements on both sides, so we begin the construction of the subgraph Ω_2 by taking j_2 separate parallel copies of the circle $\mathbf{x}_2^{(0)}$ lying in the vertex space at $v_0 \in \Gamma$ and i_2 copies of the circle $\mathbf{x}_2^{(1)}$ above $v_1 \in \Gamma$. The respective intersection numbers of $\mathbf{x}_2^{(0)}$ with z_{e_-} and $\mathbf{x}_2^{(1)}$ with z_{e_+} tell us that we have $i_2 j_2$ intersection points at each end of the cylinder C_e which is the edge space at e. Consequently we can match up these intersection points bijectively via arcs in C_e. In order to ensure that the subgraph Ω_2 created so far embeds in X, we walk round the image z_{e_-} of the boundary circle of C_e at e_-, ordering these $i_2 j_2$ points of intersection with the copies of the circles $\mathbf{x}_2^{(0)}$ which are part of our equitable set at v_0. We do the same with the corresponding $i_2 j_2$ points around z_{e_+} so that we can then run an arc in C_e between each pair of points taken in order around z_{e_-} and z_{e_+}. Then by respecting this order, the set of these arcs will embed in the cylinder C_e and thus this collection unioned with the circles, which is the subgraph Ω_2 created so far, will embed in X.

(It might be the case that the image z_{e_-} (say) in the vertex space at e_- of the relevant boundary circle of C_e does not embed, but this only occurs if z_{e_-} is a proper power w^n (for $n \geq 2$ and w primitive) in the vertex group $\mathbb{Z} \times \mathbb{Z}$ at e_-. If so then we attach the boundary circle to a curve in the torus vertex space which spirals injectively n times around the simple closed curve w before joining up the endpoints with a small arc passing through the spiral. Then using this attaching map still results in the same fundamental group for X and we can place our $i_2 j_2$ intersection points injectively around this curve as long as we avoid this small arc.)

We now take our next vertex v_2, which we assume is also joined to the base vertex v_0 by the edge f say (else we can move to the next step). In the vertex space at v_0 our equitable set is currently a single copy of $\mathbf{x}_1^{(0)}$, then j_2 copies of $\mathbf{x}_2^{(0)}$,..., j_m copies of $\mathbf{x}_m^{(0)}$ whereas for our new vertex v_2 it is just single copies each of $\mathbf{x}_1^{(2)}, \ldots, \mathbf{x}_m^{(2)}$. Let us now examine the corresponding intersection numbers; these give us new integers

$$k_1 = \#[\mathbf{x}_1^{(0)}, z_{f_-}], \quad \dots \quad , k_m = \#[\mathbf{x}_m^{(0)}, z_{f_-}] \text{ and}$$
$$l_1 = \#[\mathbf{x}_1^{(2)}, z_{f_+}], \quad \dots \quad , l_m = \#[\mathbf{x}_m^{(2)}, z_{f_+}]$$

respectively. Once again, there is a single zero in each list $k_1, \dots, k_m$ and $l_1, \dots, l_m$. Suppose that $k_r = 0$ for some $1 \le r \le m$ then permute the ls so that $l_r = 0$ also. Now we match up the sets across the edge f: assuming that $r > 1$ (as if $r = 1$ then we can proceed exactly as before), we first change our single copy of $\mathbf{x}_1^{(0)}$ at v_0 to l_1 copies of $\mathbf{x}_1^{(0)}$, then put k_1 copies of $\mathbf{x}_1^{(2)}$ at v_2. In terms of the subgraph Ω_1, what was previously just a single circle above v_1 remains as only one above v_1, but now we have l_1 circles above v_0, each giving rise to $k_1 l_1$ intersection points with the corresponding boundary circle of the edge space C_f. Moreover we also have k_1 circles above v_2 and $k_1 l_1$ intersection points with the boundary circle on the other side of C_f. Once again we pair off each of these $k_1 l_1$ intersection points on either side in an order preserving way and draw arcs between each pair as above, so that the updated graph Ω_1 embeds in X.

We now update the subgraph Ω_2, which currently has j_2 circles above v_0 corresponding to copies of the element $\mathbf{x}_2^{(0)}$ and i_2 circles above v_1 corresponding to elements $\mathbf{x}_2^{(1)}$. In order now to match the intersection numbers k_2 at v_0 and l_2 at v_2, we increase the j_2 copies of $\mathbf{x}_2^{(0)}$ at v_0 to $j_2 l_2$ copies and we put $j_2 k_2$ copies of $\mathbf{x}_2^{(2)}$ at v_2. This means that in Ω_2 we now have two lots of $j_2 k_2 l_2$ intersection points at each end of the cylinder C_f, which can be paired off and joined up by arcs using the construction above so that Ω_2 still embeds.

This is fine for the new vertex v_2 but we have changed the number of copies of $\mathbf{x}_2^{(0)}$ at v_0 from j_2 to $j_2 l_2$. However these j_2 circles had a total of $i_2 j_2$ arcs joined to the i_2 circles at v_1. Consequently we now need to change the equitable set at v_1 from i_2 copies of this element $\mathbf{x}_2^{(1)}$ to $i_2 l_2$ copies, so that the extra intersection points now above v_0 in the direction of e can all be joined up to the corresponding points above v_1, again in such away that the graph embeds.

We continue in this way where we add to Ω_s (for $s = 3, \dots, m$ in turn) the appropriate number of circles above v_2 and the correct number of multiples of our circles in Ω_s that already lie above v_0, so that we can create a bijection between these intersection points, and then replicate the same number of multiples of the circles in Ω_s above v_1. (For $s = r$ though, we have intersection number zero at v_2 which means there are no intersection points that need to be matched up. Here we merely place a single copy of $\mathbf{x}_r^{(2)}$ in Ω_r above v_2 and add no arcs.)

Now we proceed to the other vertices v_i of the graph Γ in order of our labelling, thus first working through the edges attached to v_0 and then to its descendents and so on. On adding a new edge η and new vertex v_n joined by η to a previous vertex v_p, at the sth stage we take our current copy of Ω_s and place the correct number of copies of $\mathbf{x}_s^{(n)}$ above v_n, as determined by the intersection numbers of $\mathbf{x}_s^{(p)}$ and $\mathbf{x}_s^{(n)}$ with η (or one copy if that latter number is zero, then skipping the next step), along the appropriate number of multiples of the copies of $\mathbf{x}_s^{(p)}$ already in place above v_p. We then replicate the same number of multiples of every other circle in Ω_s that has been placed so far and add the appropriate number of arcs (in fact, it is enough just to do this within the current connected component of Ω_s). We finish with the leaf vertices, thus obtaining the final graphs $\Omega_1, \ldots, \Omega_m$ which may not be connected graphs, but each of which embeds in our graph of spaces X. Thus all of their connected components, which are exactly the immersed walls obtained from our final equitable set, are embedded so we have undilated walls. Moreover this final set genuinely is an equitable set under Wise's Definition 1 for $(\mathcal{G}, \Gamma)$ because at each stage we made sure to create the necessary bijections between individual intersection points, thus also between the unions of these intersection points on either side of any edge. Finally the set is also clearly fortified and primitive, so by applying Woodhouse's criterion we conclude that $\mathcal{G}(\Gamma)$ is a virtually special group.

□

As an immediate Corollary we obtain linearity for all of these groups, because RAAGs are linear (even over $\mathbb{Z}$).

Corollary 3 *If the tubular group G is defined by the graph of groups $(\mathcal{G}, \Gamma)$ where Γ is a tree then G is linear over $\mathbb{Z}$.*

In this regard, we note the conjecture of Metaftsis, Raptis and Varsos in [21] that the fundamental group of a graph of groups, where the graph is a finite tree and the vertex groups are all finitely generated abelian, is linear. Thus this result establishes their conjecture in the case of $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups. But clearly not all tubular groups are linear.

We also note here that these groups G in Corollary 3 are all free by cyclic, by [8] Corollary 2.2, thus obtaining more free by cyclic linear groups. It is unknown whether all free by cyclic groups are linear (this is open for Example 2 below). Moreover these groups G all have linearly growing monodromy, whereas in the recent paper [3] it was shown by direct construction of special cube complexes that for each $k \in \mathbb{N}$ there is a virtually special free by cyclic group with monodromy growth polynomial of degree k.

We finish this section with a few basic examples to illustrate the different types of geometric and group theoretic behaviour that we see in tubular groups. We first note that [25] Corollary 5.10 gives a complete classification of the tubular groups acting geometrically on a CAT(0) cube complex: they have exactly one or two parallelism classes of edge groups in each vertex group and do not contain any unbalanced Baumslag - Solitar subgroup. In particular we see from Theorem 2 that there are many tubular groups which are virtually special but which do not act geometrically on any CAT(0) cube complex: namely taking $(\mathcal{G}, \Gamma)$ with Γ a tree that contains a star with three edges and such that each inclusion into the central vertex group of this star is in a different parallelism class. (As Γ is a tree, we already know that the tubular group is CAT(0) and so contains no unbalanced Baumslag - Solitar subgroups.) As for other types of behaviour, we now present two famous examples. In both cases Γ is a single point with vertex group $\langle a, b\rangle \cong \mathbb{Z}^2$ joined by two self loops.

Example 1: Wise's tubular group G, which was shown to be both non Hopfian and CAT(0) in [23], is given by the two pairs of inclusions a and a^2b^2, b and a^2b^2. Thus it is a tubular group which is CAT(0) but not virtually special. (If it were then the resulting finite index special group would be residually finite as a subgroup of a RAAG, hence so would G. However it is well known that for finitely generated groups, being residually finite implies being Hopfian.) We will see some very similar groups in the next section.

Example 2: Gersten's free by cyclic group, which was shown not to be a CAT(0) group in [14], can be expressed as a tubular group where one edge provides inclusions ab and b, and the other $a^{-1}b$ and b. Thus it is an example of a tubular group which is not CAT(0) but which is residually finite (as a free by cyclic group) and hence Hopfian, and which does not contain an unbalanced Baumslag - Solitar subgroup (for instance by [9] Proposition 2.5).

3 A 1-relator group from a tubular group, following Gardam and Woodhouse

In [13] the following family of presentations is considered:

$$\langle x, y, t | x^2 = y^2, t^{-1}x^{2q}t = x^{2p-1}y\rangle$$

where p, q are positive integers. The group $R_{p,q}$ so defined has of course a 2-generator 1-relator presentation (just eliminate y). Interestingly it is shown that $R_{p,q}$ has an index 2 subgroup $G_{p,q}$ with the following presentation:

$$G_{p,q} = \langle a, b, s, t | [a, b], s^{-1}a^q s = a^p b, t^{-1}a^q t = a^p b^{-1}\rangle$$

which we immediately recognise as a tubular group in line with the examples in Section 4. Moreover on taking $p > q \geq 1$, these are exactly the snowflake groups in [2] which have unusual Dehn functions that are greater than quadratic and so they cannot be CAT(0). Thus [13] provides the first examples of 1-relator groups which are not CAT(0) but which do not contain unbalanced Baumslag - Solitar subgroups. Also [13] Proposition 4 shows that $R_{p,q}$ does not act freely on a CAT(0) complex for $p > q$, because $G_{p,q}$ does not (by applying Wise's equitable sets condition).

But what about the 1-relator groups $R_{p,q}$ where $q \geq p \geq 1$? In this case we still have the index 2 tubular subgroup $G_{p,q}$ with presentation as above. For $p = q = 1$ [13] points out that $G_{1,1}$ is none other than Gersten's group. Thus $R_{1,1}$ is a 1-relator group that is virtually free by cyclic (indeed it is free by cyclic by K. S. Brown's criterion) with quadratic Dehn function and which acts freely on a CAT(0) complex but which is not CAT(0).

Gardam showed in his thesis [12] that the group $R_{p,q}$ for $p, q \geq 1$ is CAT(0) exactly when $q > p$. This suggests trying to mimic Wise's non Hopfian CAT(0) group construction from [23] which was Example 1 in the previous section.

Theorem 4 *For odd $q > p = 1$, the tubular group $G_{1,q}$ above is not Hopfian.*

Proof. We have

$$G_{1,q} := \langle a, b, s, t | [a, b], s^{-1}a^q s = ab, t^{-1}a^q t = ab^{-1}\rangle,$$

so can take $\theta : G_{1,q} \to G_{1,q}$ which sends a to a^q and b to b^q but fixes s and t. This preserves all three relations of $G_{1,q}$ so is a well defined homomorphism.

Now s, t, a^q, b^q are all in the image of θ, thus so too is ab and ab^{-1}. As q is odd, we also get a and b so θ is onto.

To show θ is not injective, we apply the usual properties of HNN extensions as in [23] and even in the original construction of Baumslag - Solitar groups. The commutator

$$[s^{-1}as, ab^{-1}] = s^{-1}asab^{-1}s^{-1}a^{-1}sa^{-1}b$$

is a non trivial element of $G_{1,q}$ by Britton's Lemma for multiple HNN extensions as there are no pinches, but it maps to the identity as $\theta(s^{-1}as) = ab$.
□

Corollary 5 *The 1-relator group $R_{1,q}$ for odd $q \geq 3$ is CAT(0) but contains the index 2 subgroup $G_{1,q}$ which is not residually finite, thus nor is $R_{1,q}$.*

In particular $R_{1,q}$ is not virtually special. We believe this is the first known 1-relator group which is CAT(0) but not residually finite.

We now consider the question of when the group $G_{p,q}$ (equivalently $R_{p,q}$) is residually finite. Here we will not insist that p, q are positive, rather we will allow them to be any non zero integers. By Malce'ev's result that a finitely generated residually finite group is Hopfian, we know that any element in the kernel of a surjective endomorphism must lie in the intersection of all finite index subgroups, motivating the next proof.

Theorem 6 *The group $G_{p,q}$ is not residually finite if q does not divide $2p$.*

Proof. Let $h = (2p, q)$ be the highest common factor of $2p$ and q, so that $0 < h < q$. Consider the element $x = [s^{-1}a^h s, a^p b^{-1}]$ which, as in Theorem 4 above, has no pinches and so is a non trivial element of $G_{p,q}$.

Now take any homomorphism from $G_{p,q}$ to a finite group F and let r be the order of (the image of) a in F, so that a^q has order d where $d = r/(r, q)$. This means that both $a^p b$ and $a^p b^{-1}$ have order d too, but a and b still commute in F so that $e = a^{2pd} = b^{2d}$. Hence r divides $2pd$ and so (r, q) divides $2p$.

Consequently (r, q) divides both $2p$ and q, hence h also. Thus there will exist integers u, v with $ru + vq = h$, so that in F we have $s^{-1}a^h s = s^{-1}a^{ru+vq}s = (s^{-1}a^q s)^v$. Hence $s^{-1}a^h s$ is a power of $s^{-1}a^q s = a^p b$ in F and will therefore commute with any elements that commute with $a^p b$, in particular a, b and $a^p b^{-1}$. Thus the element $x \in G_{p,q}$ above is non trivial, but trivial in any finite quotient.

□

Theorem 3.7 of [19] shows that a tubular group (or indeed the fundamental group of any graph of finitely generated free abelian groups with infinite cyclic edge groups) is residually finite if each edge inclusion sends the generator of the edge group $\mathbb{Z}$ to a primitive element of the corresponding vertex group. This allows us to obtain a complete characterisation of the residually finite groups in the $G_{p,q}$ family, and hence in the $R_{p,q}$ family too.

Corollary 7 *The group $G_{p,q}$ is residually finite if and only if q divides $2p$.*

Proof. The elements a^pb and a^pb^{-1} are certainly primitive in the edge group $\langle a, b\rangle$, so if $q = \pm 1$ then the above result applies. Otherwise first suppose that q divides p. Then we follow the idea in [10] Proposition 3.1. We have a homomorphism from $G_{p,q}$ to the cyclic group C_q given by sending a to 1 and b, s, t to zero. Now $G_{p,q}$ is a tubular group and thus acts on its Bass - Serre tree T, so the kernel K has finite index q in $G_{p,q}$ and also acts on T with $\mathbb{Z}^2$ vertex stabilisers and $\mathbb{Z}$ edge stabilisers. This gives rise to a decomposition of K as a tubular group, with edge groups $E \cap K$ and vertex groups $V \cap K$ where E and V are the edge and vertex groups for G on T, thus are conjugates in G of $\langle a, b\rangle$ and a^q, a^pb, a^pb^{-1} respectively. Now $\langle a, b\rangle \cap K = \langle a^q, b\rangle$ and as K is normal, this will hold for any conjugate of $\langle a, b\rangle$ too. Thus now the edge inclusions of a^q, a^pb, a^pb^{-1} are all primitive elements in the vertex groups of K.

If q does not divide p, so that $q = 2m$ is even with m dividing p, we proceed in exactly the same way but now we use the homomorphism from $G_{p,q}$ to C_q sending a to 1, b to p and s, t to 0.

□

We note that the forthcoming paper [18] characterises exactly when a tubular group is residually finite.

Finally we can also answer Question 20 in [7] from 2010, which was put forward to suggest that we knew very little about how widespread residually finite 2-generator 1-relator groups might be. It says:

Let G be a group with a 2-generator 1-relator presentation where the relator is not a proper power. Suppose that G is residually finite then does G

have a finite index subgroup H which is an ascending HNN extension of a finitely generated free group?

Using the above constructions, we have:

Corollary 8 *The group $R_{p,q}$ above where $p > q \geq 1$ and q divides $2p$ is residually finite, but no finite index subgroup of $R_{p,q}$ is an ascending HNN extension of a finitely generated free group.*

Proof. For these values of p, q we know from above that $R_{p,q}$ is residually finite but has a Dehn function which is bigger than quadratic. By [4] all free by cyclic groups $F_n \rtimes_\alpha \mathbb{Z}$ have at most quadratic Dehn function and this is a quasi isometry invariant, so $R_{p,q}$ has no finite index subgroup of this form either.

However this does not rule out $R_{p,q}$ having a finite index subgroup which is is a strictly ascending HNN extension of a finitely generated free group (as such groups can contain Baumslag - Solitar subgroups of the form $BS(1, q)$ and so have exponential Dehn function). For this, we note that the corresponding tubular group $G_{p,q}$ will have such a finite index subgroup H too, by [6] Proposition 4.3 (iii). Now H will also be a tubular group, but the contrapositive of [11] Corollary 2.12 implies that a tubular group cannot be a strictly ascending HNN extension of any finitely generated group because the graph of groups decomposition defining the tubular group is not an ascending HNN extension.

$\square$

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